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INDUCTIVE ALGEBRAS AND HOMOGENEOUS SHIFTS

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Abstract. Inductive algebras for the irreducible unitary representations of the universal cover of the group of unimodular two by two matrices are classified. The classification of homogeneous shift operators is obtained as a direct consequence. This gives a new approach to the results of Bagchi and Misra.

1. Introduction

Let G be a separable locally compact group and let R be a strongly continuous representation of G on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. An *inductive algebra* is a strong-operator closed abelian sub-algebra of $\mathcal{B}(\mathcal{H})$ that is normalized by $R(G)$. If we wish to emphasize the dependence on R, we use the term R-inductive algebra. The inductive algebras for the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ were classified in [5]. In this work, we classify the inductive algebras for the irreducible unitary representations of the universal cover of $\mathrm{SL}(2,\mathbb{R})$. We use this result to give another proof of the classification of the homogeneous shifts originally obtained by Bagchi and Misra [1].

Let $\mathbb{D}$ and $\mathbb{T}$ denote the unit disc and the unit circle in the complex plane $\mathbb{C}$ respectively. Let Möb denote the Möbius group (the group of biholomorphic automorphisms of $\mathbb{D}$). Thus $\text{Möb} = \{\varphi_{\alpha,\beta} \mid \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha\frac{z-\beta}{1-\overline{\beta}z}, \qquad z \in \mathbb{D}.$$

The correspondence $(\alpha,\beta) \mapsto \varphi_{\alpha,\beta}$ is bijective and identifies Möb with the manifold $\mathbb{T}\times\mathbb{D}$. With this differential structure, Möb is a Lie group isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for those $\varphi \in$ Möb which are holomorphic on the spectrum of T. The homogeneous weighted shift operators (the homogeneous shifts, for brevity) were classified in [1].

It was noticed by Gadadhar Misra that the operators which (topologically) generate the inductive algebras found in [5] are weighted shifts whose weight-sequences were the same as *some* of the operators found in [1]. As explained in [1], every homogeneous shift is (non-uniquely) associated to a *simple* representation of the universal cover $\widetilde{\text{Möb}}$ of Möb. Furthermore, it turns out that the closure in the strong-operator topology of the sub-algebra generated by the set $\{\varphi(T) \mid \varphi \in \text{Möb}\}$ is inductive for this representation (see Prop. 4.1). Thus a classification of the inductive algebras for the simple representations of

$\widetilde{\text{Möb}}$ would yield *all* the operators that were found in [1], and in fact an independent proof of the classification (Theorem 5.2 in [1]).

Here, we find the inductive algebras only for the irreducible unitary representations of $\widetilde{\text{Möb}}$. That still leaves the reducible simple representations, i.e. those of the form $D^-_{2-\lambda} \oplus D^+_\lambda$, $0 < \lambda < 2$. The *assumption* that T is a (scalar) shift allows us to patch together information contained in the inductive algebras for $D^-_{2-\lambda}$ and D^+_λ to determine the homogeneous operators associated to $D^-_{2-\lambda} \oplus D^+_\lambda$.

Every homogeneous operator is a block shift [1], but the problem of classifying homogeneous operators remains open. This work addresses the case where the blocks are one dimensional. Extending it elegantly to the case of higher dimensional blocks would require the classification of inductive algebras for reducible representations (the trick for treating $D^-_{2-\lambda} \oplus D^+_\lambda$ does not work in general). Classifying inductive algebras for reducible representations appears to be a much harder problem (akin to classifying all abelian sub-algebras of a matrix algebra).

The homogeneous operators belonging to a Cowen-Douglas class were recently classified (see [6], [2], [3]).

In section 2 we recall the relevant notation from [1] and [5] and describe the irreducible representations of $\widetilde{\text{Möb}}$ in a convenient way. In section 3, we determine the inductive algebras for each of these representations. In section 4, we use the results of section 3 to classify the homogeneous shifts.

2. The unitary dual of $\widetilde{\text{Möb}}$

Henceforth let $G = \widetilde{\text{Möb}}$ and let $\pi : G \to \text{Möb}$ denote the universal covering map. If $g \in G$ let φ_g denote $\pi(g)$ thought of as a function $\mathbb{D} \to \mathbb{D}$. Since G is connected and simply-connected, for each $\eta \in \mathbb{C}$ there is a unique smooth branch of the function $(\varphi'_{g^{-1}}(z))^\eta$ such that $(\varphi'_1(z))^\eta = 1$.

Recall that if $\varphi \in \text{Möb}$ and $\varphi^*(z) := \overline{\varphi(\overline{z})}$ then $\varphi \mapsto \varphi^*$ is an automorphism of Möb (see equation (2.1) in [1]). We let $g \mapsto g^*$ be the lift of this automorphism to G. Then

$$\varphi_{g^*} = \varphi_g^*.$$

If R is a representation of G, then $R^\#$ is the representation defined by

$$R^\#(g) = R(g^*), \qquad g \in G.$$

This notation is an extension of equation (2.4) in [1].

Let $\lambda \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $I \in \{\mathbb{Z}, \mathbb{Z}^+\}$, and assume $\mu = 0$ if $I = \mathbb{Z}^+$. Let $\mathcal{M}(\mathbb{Z})$ (resp. $\mathcal{M}(\mathbb{Z}^+)$) denote the functions which have a holomorphic extension to some neighborhood of $\mathbb{T}$ (resp. $\overline{\mathbb{D}}$).

For $g \in G$, define $R(g) = R_{\lambda,\mu}(g) : \mathcal{M}(I) \to \mathcal{M}(I)$ by

$$\begin{aligned}(R(g)F)(z) &:= (\varphi'_{g^{-1}}(z))^{\lambda/2} \left|\varphi'_{g^{-1}}(z)\right|^\mu F(\varphi_{g^{-1}}(z)) \\ &= (\varphi'_{g^{-1}}(z))^{(\lambda+\mu)/2} \overline{(\varphi'_{g^{-1}}(z))^{\mu/2}} F(\varphi_{g^{-1}}(z)).\end{aligned}$$

Then $g \mapsto R(g)$ is a representation of G on $\mathcal{M}(I)$.

Let $f_n(z) = z^n$. If we give $\mathcal{M}(I)$ the C^∞ (Frechet) topology, then the linear span of $\{f_n\}_{n\in I}$ is dense in $\mathcal{M}(I)$. Let

$$\|f_n\|^2 = \frac{\Gamma(1-\mu+n)}{\Gamma(\lambda+\overline{\mu}+n)}, \qquad n \in I, \tag{1}$$

when λ and μ are such that the expression on the right is real and positive. Extend $\|\cdot\|$ to $\operatorname{span}\{f_n\}_{n\in I}$ as a norm. Since $\|f_n\|$ grows at most like a polynomial in $|n|$ as $|n| \to \infty$, it follows that $\|\cdot\|$ is uniformly continuous, and thus extends to $\mathcal{M}(I)$. Let $\mathcal{H} = \mathcal{H}^{\lambda,\mu}$ be the completion of $\mathcal{M}(I)$ under this norm. Then R extends to a unitary representation of G on $\mathcal{H}$. Pukánszky [4] showed that every irreducible unitary representation of G is unitarily equivalent either to a representation of this type or to the composition of one with the $*$-automorphism. We recall his taxonomy in terms of our parameters (I, λ, μ):

Holomorphic discrete series: $D^+_\lambda = R_{\lambda,0}$ where $I = \mathbb{Z}^+$ and $\lambda > 0$.
Anti-holomorphic discrete series: $D^-_\lambda = (D^+_\lambda)^\#$, $\lambda > 0$.
Principal series: $R_{\lambda,\mu}$, $I = \mathbb{Z}$, $\lambda \in (-1, 1]$ and $\operatorname{Re}\mu = \frac{1-\lambda}{2}$.
Complementary series: $R_{\lambda,\mu}$, $I = \mathbb{Z}$, $\lambda \in (-1, 1)$ and $\mu \in (0, 1) \cap (-\lambda, 1-\lambda)$.

Within the principal and complementary series, there are unitary equivalences between $R_{\lambda,\mu}$ and $R_{\lambda,1-\lambda-\mu}$ and $R_{1,0} \cong D^+_1 \oplus D^-_1$. Otherwise, these representations are irreducible, inequivalent and cover the unitary dual of G. We remark that the usage "Discrete series" here is a historical accident, and in fact $D^\pm_\lambda$ do not embed in $L^2(G)$.

3. The inductive algebras

Let R be one of the irreducible unitary representations $R_{\lambda,\mu}$ introduced in section 2. For $g \in G$, define $\tilde{\kappa}(g) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $T \mapsto R(g)TR(g)^{-1}$. If $g \in \ker(\pi)$ then $R(g)$ is a scalar, so $\tilde{\kappa}(g)$ is trivial. So $\tilde{\kappa}$ descends to a representation κ of Möb. The remarks about κ at the beginning of section 3 in [5] continue to be valid.

Let $K = \{\varphi_{\alpha,0} \mid \alpha \in \mathbb{T}\} \subseteq$ Möb. Then K is a maximal compact subgroup. For $l \in \mathbb{Z}$, let $\chi_l : K \to \mathbb{C}^*$ be defined by $\chi_l(\varphi_{\alpha,0}) = \alpha^l$. Then $\widehat{K} = \{\chi_l\}_{l\in\mathbb{Z}}$.

Let $\mathfrak{g}$ denote the Lie algebra of Möb. We let exp denote the exponential map from $\mathfrak{g}$ to any of its associated Lie groups. The precise group will be clear from context. Let $h, L, M \in \mathfrak{g}$ be such that

$$\exp th = \varphi_{e^{2it},0},$$
$$\exp tL = \varphi_{1,-\tanh t},$$
$$\exp tM = \varphi_{1,-i\tanh t},$$

and $e = \frac{1}{2}(L - iM)$ and $f = \frac{1}{2}(L + iM)$. Then $h, e, f \in \mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C})$.

Observe that

$$\{F \in \mathcal{H} \mid R(\exp(th))F = e^{-i(2n+\lambda)t}F\} = \mathbb{C}f_n, \qquad n \in I.$$

Differentiating along one-parameter subgroups (and then forming complex linear combinations) gives

$$R(e)f_n = \begin{cases} (\mu - n)\, f_{n-1}, & n \in I \cap (I+1), \\ 0 & \text{otherwise}, \end{cases}$$
$$R(f)f_n = (\lambda + \mu + n)\, f_{n+1}, \qquad n \in I.$$

Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an R-inductive algebra. For $m \in \mathbb{Z}$, define the vector spaces

$$\mathcal{A}_m = \{T \in \mathcal{A} \mid \kappa(g)T = \chi_m(g)T,\ g \in K\}.$$

Then by the Peter-Weyl theorem

$$\bigoplus_{m\in\mathbb{Z}} \mathcal{A}_m$$

is sequentially dense in $\mathcal{A}$ (see section 3 of [5]).

If $T \in \mathcal{A}_m$ then

$$\begin{aligned} R(\exp(th))Tf_n &= \kappa(\exp(th))TR(\exp(th))f_n \\ &= \chi_m(\exp(th))TR(\exp(th))f_n \\ &= e^{i(2m-2n-\lambda)t}Tf_n, \end{aligned}$$

so $Tf_n \in \mathbb{C}f_{n-m}$. So

$$Tf_n = \begin{cases} a_n f_{n-m} & n \in I \cap (I+m), \\ 0 & \text{otherwise}, \end{cases}$$

for some $a_n \in \mathbb{C}$. A similar calculation shows that conversely if T has this form, then $T \in \mathcal{A}_m$.

Let $\mathcal{H}^\infty$ and $\mathcal{A}^\infty$ denote the smooth vectors in $\mathcal{H}$ and $\mathcal{A}$ respectively. Recall that v is smooth if the orbit map $g \mapsto g \cdot v$ is a smooth mapping on G. We recall the following facts from section 3 of [5], and use them without mention in the calculations that follow.

- $\mathcal{A}^\infty$ is invariant under $\kappa(G)$ as well as $\kappa(\mathfrak{g})$.
- If $T \in \mathcal{A}^\infty$ and $F \in \mathcal{H}^\infty$ then $TF \in \mathcal{H}^\infty$.
- $\mathcal{A}_m \cap \mathcal{A}^\infty$ is sequentially dense in $\mathcal{A}_m$.

For $T \in \mathcal{A}^\infty$, define $T_e = [R(e), T]$ and $T_f = [R(f), T]$. Then $T_e, T_f \in \mathcal{A}^\infty$. If $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$, then $T_e \in \mathcal{A}_{m+1}$ and $T_f \in \mathcal{A}_{m-1}$. Moreover,

$$\begin{aligned} T_e f_n &= ((\mu - n + m)a_n - (\mu - n)a_{n-1})f_{n-m-1}, \qquad n \in (I+1) \cap (I+m+1), \\ T_f f_n &= ((\lambda + \mu + n - m)a_n - (\lambda + \mu + n)a_{n+1})f_{n-m+1}, \qquad n \in I \cap (I+m). \end{aligned} \tag{2}$$

For other values of n these formulas degenerate, and we will have to consider cases, when the need arises.

Lemma 3.1. $\mathcal{A}_0 = \mathbb{C}\mathrm{I}$

Proof. It is clear that $\mathbb{C}\mathrm{I} \subseteq \mathcal{A}_0$. Let $T \in \mathcal{A}_0 \cap \mathcal{A}^\infty$. Then

$$\begin{aligned} 0 &= [T, T_e] f_n \\ &= -(\mu - n)(a_{n-1} - a_n)^2 f_{n-1} \quad \text{for all } n \in I+1. \end{aligned}$$

Since $\mu \notin I+1$ for any of the representations under consideration, it follows that $a_{n+1} = a_n$ for all $n \in I$. So $\mathcal{A}_0 \cap \mathcal{A}^\infty = \mathbb{C}I$, which is dense, and being finite dimensional, closed in $\mathcal{A}_0$. □

Lemma 3.2. *For $m \neq 0$, $\mathcal{A}_{m-1} = 0 \implies \mathcal{A}_m = 0$.*

Proof. Suppose $m \neq 0$, $\mathcal{A}_{m-1} = 0$ and $T \in \mathcal{A}_m$. Write

$$T f_n = a_n f_{n-m}, \qquad n \in I \cap (I+m).$$

Since $T_f \in \mathcal{A}_{m-1}$, we have

$$(\lambda + \mu + n - m) a_n - (\lambda + \mu + n) a_{n+1} = 0, \qquad n \in I \cap (I+m). \tag{3}$$

Since $-(\lambda + \mu) \notin I$ for any of the representations under consideration, it follows that if $a_n = 0$ for some n then $a_n = 0$ for all n.

Since T commutes with T_e, we have

$$\begin{aligned} 0 &= [T, T_e] f_n \\ &= (-(\mu - n) a_{n-1} a_{n-m-1} + 2(\mu - n + m) a_n a_{n-m-1} - (\mu - n + 2m) a_n a_{n-m}) f_{n-2m-1} \\ &= \frac{\begin{pmatrix} -(\mu - n)(\lambda + \mu + n - 1) \\ +2(\mu - n + m)(\lambda + \mu + n - m - 1) \\ -(\mu - n + 2m)(\lambda + \mu + n - 2m - 1) \end{pmatrix}}{(\lambda + \mu + n - m - 1)} a_n a_{n-m-1} f_{n-2m-1}, \qquad \text{(for large } n\text{, by (3))}, \\ &= \frac{2m^2}{(\lambda + \mu + n - m - 1)} a_n a_{n-m-1} f_{n-2m-1}. \end{aligned}$$

So for large n either $a_n = 0$ or $a_{n-m-1} = 0$. Now (3) implies $a_n = 0$ for all $n \in I \cap (I+m)$. □

Lemma 3.3. *For $m \neq 0$, $\mathcal{A}_{m+1} = 0 \implies \mathcal{A}_m = 0$.*

Proof. Suppose $m \neq 0$, $\mathcal{A}_{m+1} = 0$ and $T \in \mathcal{A}_m$. Write

$$T f_n = a_n f_{n-m}, \qquad n \in I \cap (I+m).$$

Since $T_e \in \mathcal{A}_{m+1}$, we have

$$(\mu - n + m) a_n - (\mu - n) a_{n-1} = 0, \qquad n \in (I+1) \cap (I+m+1). \tag{4}$$

Since $\mu \notin \mathbb{Z}$ for any of the representations under consideration, it follows that if $a_n = 0$ for some n then $a_n = 0$ for all n.

Since T commutes with T_f, we have

$$\begin{aligned}
0 &= [T, T_f]f_n \\
&= (-(\lambda+\mu+n)a_{n+1}a_{n-m+1} + 2(\lambda+\mu+n-m)a_n a_{n-m+1} - (\lambda+\mu+n-2m)a_n a_{n-m})f_{n-2m+1} \\
&= \frac{\begin{pmatrix} -(\lambda+\mu+n)(\mu-n-1) \\ +2(\lambda+\mu+n-m)(\mu-n+m-1) \\ -(\lambda+\mu+n-2m)(\mu-n+2m-1) \end{pmatrix}}{\mu-n+m-1} a_n a_{n-m+1} f_{n-2m+1}, \qquad \text{(for large } n \text{ by (4))}, \\
&= \frac{2m^2 a_n a_{n-m+1}}{\mu-n+m-1} f_{n-2m+1},
\end{aligned}$$

So for large n, either $a_n = 0$ or $a_{n-m+1} = 0$. Now (4) implies $a_n = 0$ for all $n \in I \cap (I + m)$. □

One possibility is that $\mathcal{A} = \mathbb{C}\mathrm{I}$. Henceforth let us assume that $\mathcal{A} \neq \mathbb{C}\mathrm{I}$. Then by the Peter-Weyl theorem there exists $m \neq 0$ such that $\mathcal{A}_m \neq 0$. By Lemmas 3.2 and 3.3, it follows that either $\mathcal{A}_1 \neq 0$ or $\mathcal{A}_{-1} \neq 0$.

To analyze the situation further, it is now best to consider cases.

Case $I = \mathbb{Z}^+$ ($\mu = 0$):

Suppose $T \in \mathcal{A}_{-1} \cap \mathcal{A}^\infty$. Then $Tf_n = a_n f_{n+1}$ for some $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$. Since $T_e \in \mathcal{A}_0$, by Lemma 3.1 there exists $b \in \mathbb{C}$ such that $T_e = -b\mathrm{I}$. We compute

$$T_e f_0 = R(e)Tf_0 = -a_0 f_0.$$

So $a_0 = b$. Now (2) gives $a_n = b$, $n = 0, 1, 2, \ldots$. So T is a multiple of T_1 where

$$T_1 f_n = f_{n+1}, \qquad n = 0, 1, 2, \ldots. \tag{5}$$

Suppose $T \in \mathcal{A}_1 \cap \mathcal{A}^\infty$. Then $Tf_0 = 0$ and $Tf_n = a_n f_{n-1}$ for some $a_n \in \mathbb{C}$, $n = 1, 2, \ldots$. Since $T_f \in \mathcal{A}_0$, by Lemma 3.1 there exists $b \in \mathbb{C}$ such that $T_f = -b\mathrm{I}$. We compute

$$T_f f_0 = -TR(f)f_0 = -\lambda a_1 f_0.$$

So $a_1 = b/\lambda$. Now (2) gives

$$a_n = \frac{nb}{\lambda+n-1}, \qquad n = 1, 2, \ldots.$$

So T is a multiple of T_1^*.

Since T_1 and T_1^* don't commute, it follows that at most one of $\mathcal{A}_{-1}$ and $\mathcal{A}_1$ is nonzero.

Suppose $\mathcal{A}_{-1} \neq 0$. Then $\mathcal{A}_1 = 0$ and so $\mathcal{A}_m = 0$ for $m = 1, 2, \ldots$ by Lemma 3.3. Also $\mathcal{A}_{-1} = \mathbb{C}T_1$ by the previous calculation (and the fact that finite dimensional subspaces are closed). Let $m > 0$ and $T \in \mathcal{A}_{-m}$. The equation $[T, T_1] = 0$ implies $T \in \mathbb{C}T_1^m$. By Lemma 3.2, $\mathcal{A}_{-m} \neq 0$, so in fact $\mathcal{A}_{-m} = \mathbb{C}T_1^m$. So $\mathcal{A}$ is the strong-operator closure of the algebra generated by T_1.

Suppose $\mathcal{A}_1 \neq 0$. Then $\mathcal{A}_1 = 0$, and so $\mathcal{A}_m = 0$ for $m = -1, -2, \ldots$ by Lemma 3.2. Also $\mathcal{A}_1 = \mathbb{C}T_1^*$ by the previous calculation. Let $m > 0$ and $T \in \mathcal{A}_m$. The equation $[T, T_1^*] = 0$ implies $T \in \mathbb{C}(T_1^*)^m$. By Lemma 3.3, $\mathcal{A}_m \neq 0$, so in fact $\mathcal{A}_m = \mathbb{C}(T_1^*)^m$. So $\mathcal{A}$ is the strong-operator closure of the algebra generated by T_1^*.

Case $I = \mathbb{Z}$:

Suppose $T \in \mathcal{A}_{-1}$. Then $Tf_n = a_n f_{n+1}$ for some $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. Since $T_e \in \mathcal{A}_0$, by Lemma 3.1 there exists $b \in \mathbb{C}$ such that $T_e = -b\mathrm{I}$. Together with (2), this reads

$$(\mu - n - 1)a_n - (\mu - n)a_{n-1} = -b,$$

so there exists $a \in \mathbb{C}$ such that

$$a_n = \frac{a - bn}{\mu - n - 1}.$$

Since T commutes with T_f, we have

$$\begin{aligned} 0 &= [T, T_f]f_n \\ &= (-(\lambda + \mu + n)a_{n+1}a_{n+2} + 2(\lambda + \mu + n + 1)a_n a_{n+2} - (\lambda + \mu + n + 2)a_n a_{n+1})f_{n+3} \\ \therefore \quad 0 &= (a + b(1 - \mu))(a + b(\lambda + \mu)). \end{aligned}$$

Therefore either $a = b(\mu - 1)$ or $a = -b(\lambda + \mu)$. So T is a multiple of either T_2 or T_3 where

$$\begin{aligned} T_2 f_n &= f_{n+1}, \qquad n \in \mathbb{Z}, \\ T_3 f_n &= \frac{\lambda + \mu + n}{n + 1 - \mu} f_{n+1}, \qquad n \in \mathbb{Z}. \end{aligned} \tag{6}$$

Suppose $T \in \mathcal{A}_1$. Then $Tf_n = a_n f_{n-1}$ for some $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. Since $T_f \in \mathcal{A}_0$, by Lemma 3.1 there exists $b \in \mathbb{C}$ such that $T_f = -b\mathrm{I}$. Together with (2), this reads

$$(\lambda + \mu + n - 1)a_n - (\lambda + \mu + n)a_{n+1} = -b,$$

so there exists $a \in \mathbb{C}$ such that

$$a_n = \frac{a - bn}{\lambda + \mu + n - 1}.$$

Since T commutes with T_e, we have

$$\begin{aligned} 0 &= [T, T_e]f_n \\ &= (-(\mu - n)a_{n-1}a_{n-2} + 2(\mu - n + 1)a_n a_{n-2} - (\mu - n + 2)a_n a_{n-1})f_{n-3} \\ \therefore \quad 0 &= (a + b(\lambda + \mu - 1))(a - b\mu). \end{aligned}$$

Therefore either $a = b(1 - \lambda - \mu)$ or $a = b\mu$. So T is a multiple of either T_2^{-1} or T_3^{-1}.

Suppose $\mathcal{A}_{-1} \neq 0$. Then by the previous calculation, $\mathcal{A}_{-1} = \mathbb{C}T_2$ or $\mathcal{A}_{-1} = \mathbb{C}T_3$. Suppose $\mathcal{A}_{-1} = \mathbb{C}T_2$. Let $m > 0$ and $T \in \mathcal{A}_{-m}$. The equation $[T, T_2] = 0$ implies $T \in \mathbb{C}T_2^m$. By Lemma 3.2, $\mathcal{A}_{-m} \neq 0$, so in fact $\mathcal{A}_{-m} = \mathbb{C}T_2^m$. Likewise, $\mathcal{A}_{-1} = \mathbb{C}T_3$ implies $\mathcal{A}_{-m} = \mathbb{C}T_3^m$ for all $m > 0$.

A similar argument shows that if $\mathcal{A}_1 \neq 0$ then either $\mathcal{A}_m = \mathbb{C}T_2^{-m}$ for all $m > 0$ or $\mathcal{A}_m = \mathbb{C}T_3^{-m}$ for all $m > 0$.

Since T_2 and T_3 don't commute unless they are equal (which occurs precisely when $\mu = (1-\lambda)/2$), it now follows that $\mathcal{A}$ is the strong operator closure of one of the following algebras: $\mathbb{C}[T_2]$, $\mathbb{C}[T_3]$, $\mathbb{C}[T_2^{-1}]$, $\mathbb{C}[T_3^{-1}]$, $\mathbb{C}[T_2, T_2^{-1}]$ and $\mathbb{C}[T_3, T_3^{-1}]$.

Case $R = D_\lambda^-$: Now $\mathcal{A}$ is D_λ^--inductive iff it is D_λ^+-inductive. However, $\mathcal{A}_m$ and $\mathcal{A}_{-m}$ are interchanged. To see this, let $(\kappa^+, \mathcal{A}^+)$ and $(\kappa^-, \mathcal{A}^-)$ denote the conjugation action of Möb on $\mathcal{A}$ under D_λ^+ and D_λ^- respectively. Then

$$\begin{aligned}
\mathcal{A}_m^- &= \{T \in \mathcal{A} \mid \kappa^-(g)T = \chi_m(g)T,\ g \in K\} \\
&= \{T \in \mathcal{A} \mid \kappa^+(g^*)T = \chi_m(g)T,\ g \in K\} \\
&= \{T \in \mathcal{A} \mid \kappa^+(g)T = \chi_m(g^*)T,\ g \in K\} \\
&= \{T \in \mathcal{A} \mid \kappa^+(g)T = \chi_{-m}(g)T,\ g \in K\} \\
&= \mathcal{A}_{-m}^+
\end{aligned}$$

In particular, $\mathcal{A}_{-1}^- = \mathcal{A}_1^+ = \mathbb{C}T_1^*$, where (we recall)

$$T_1^* f_n = \frac{n}{\lambda + n - 1} f_{n-1}, \qquad n = 1, 2, \ldots. \tag{7}$$

4. The homogeneous shifts

Let $T \in \mathcal{B}(\mathcal{H})$ be homogeneous. Recall (Definition 2.1 in [1]) that a representation R of G on $\mathcal{H}$ is *associated* with T if

$$\varphi_g(T) = R(g)^{-1} T R(g), \qquad g \in G. \tag{8}$$

Let $\mathcal{A}$ be the strong-operator closure of the subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{\varphi(T) \mid \varphi \in \text{Möb}\}$.

Proposition 4.1. *The algebra $\mathcal{A}$ is R-inductive and $T \in \mathcal{A}_{-1}$.*

Proof. Since all elements of $\mathcal{A}$ are functions of a single operator, it is abelian. Furthermore, (8) shows that $\mathcal{A}$ is normalized by $R(G)$. So $\mathcal{A}$ is an R-inductive algebra. If $\tilde{\kappa}$ and κ are the conjugation representations of G and Möb on $\mathcal{A}$, then

$$\begin{aligned}
\tilde{\kappa}(g)T = R(g)TR(g)^{-1} &= \varphi_{g^{-1}}(T), && g \in G, \\
\therefore \quad \kappa(\varphi)T &= \varphi^{-1}(T), && \varphi \in \text{Möb}, \\
\therefore \quad \kappa(\varphi_{\alpha,0})T &= \varphi_{\alpha,0}^{-1}(T) = \alpha^{-1}T, && \alpha \in \mathbb{T},
\end{aligned}$$

so $T \in \mathcal{A}_{-1}$. □

Let $I \in \{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-\}$. Recall that an operator T on a Hilbert space $\mathcal{H}$ is called a *weighted shift* with weight sequence w_n, $n \in I$, if there is a distinguished orthonormal basis x_n, $n \in I$ such that $Tx_n = w_n x_{n+1}$ for all $n \in I$. The weighted shift is called a *bilateral shift*, *forward shift* or *backward shift* according as $I = \mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$.

It was shown in [1] that if $T \in \mathcal{B}(\mathcal{H})$ is a homogeneous weighted shift, then there is a unitary representation R of G on $\mathcal{H}$ associated to T and the h-isotypic subspaces of $\mathcal{H}$ are precisely $\mathbb{C}x_n$, i.e., for each $n \in I$ there exists $\lambda_n \in \mathbb{R}$ such that

$$\{F \in \mathcal{H} \mid R(\exp(th))F = e^{-\lambda_n t}F,\, t \in \mathbb{R}\} = \mathbb{C}x_n.$$

Further, it was shown that either R is irreducible, or has the decomposition $R \cong D^-_{2-\lambda} \oplus D^+_\lambda$.

Here, we will give different proofs of Lemma 5.1 and Theorem 5.2 from [1].

First observe that T_1, T_1^*, T_2, and T_3 are homogeneous by Theorem 2.3 in [1].

Remark 4.2. If both T and cT are associated to the same representation, then $\varphi(cT) = c\varphi(T)$ for all $\varphi \in$ Möb and so $c = 1$.

Suppose T is a homogeneous (weighted) shift and R is associated to T.

Assume first that R is irreducible.

If $R = D^+_\lambda$ is in the holomorphic discrete series, then T is a multiple of T_1 by (5). So, by Remark 4.2, $T = T_1$. If

$$x_n = \sqrt{\frac{\Gamma(\lambda+n)}{\Gamma(1+n)}}f_n, \qquad n \in \mathbb{Z}^+,$$

then x_n is an orthonormal basis of $\mathcal{H}^{\lambda,0}$ and in terms of this basis,

$$\begin{aligned} T_1x_n &= \sqrt{\frac{\Gamma(\lambda+n)}{\Gamma(1+n)}}T_1 f_n \\ &= \sqrt{\frac{\Gamma(\lambda+n)}{\Gamma(1+n)}}f_{n+1} \\ &= \sqrt{\frac{\Gamma(\lambda+n)}{\Gamma(1+n)}}\sqrt{\frac{\Gamma(2+n)}{\Gamma(\lambda+n+1)}}x_{n+1} \\ &= \sqrt{\frac{1+n}{\lambda+n}}x_n, \end{aligned}$$

so the weight sequence is $w_n = \sqrt{\frac{1+n}{\lambda+n}}$, $n \in \mathbb{Z}^+$.

If $R = D^-_\lambda$ is in the anti-holomorphic discrete series, then $T = T_1^*$ by (7) and Remark 4.2. If

$$x_n = \sqrt{\frac{\Gamma(\lambda-n)}{\Gamma(1-n)}}f_{-n}, \qquad n \in \mathbb{Z}^-,$$

then x_n is an orthonormal basis of $\mathcal{H}^{\lambda,0}$ and in terms of this basis,

$$\begin{aligned}
T_1^* x_n &= \sqrt{\frac{\Gamma(\lambda-n)}{\Gamma(1-n)}} T_1^* f_{-n} \\
&= \sqrt{\frac{\Gamma(\lambda-n)}{\Gamma(1-n)}} \frac{-n}{\lambda-n-1} f_{-n-1} \\
&= \sqrt{\frac{\Gamma(\lambda-n)}{\Gamma(1-n)}} \sqrt{\frac{\Gamma(-n)}{\Gamma(\lambda-n-1)}} \frac{-n}{\lambda-n-1} x_{n+1} \\
&= \sqrt{\frac{n}{1-\lambda+n}} x_{n+1}
\end{aligned}$$

so the weight sequence is $w_n = \sqrt{\frac{n}{1-\lambda+n}}$, $n \in \mathbb{Z}^-$.

If $R = R_{\lambda,\mu}$ is in the principal series, then $T \in \{T_2, T_3\}$ by (6) and Remark 4.2. Also $x_n = f_n$, $n \in \mathbb{Z}$ is already an orthonormal basis of $\mathcal{H}^{\lambda,\mu}$. So the weight sequence is either $w_n = 1$, $n \in \mathbb{Z}$, or $w_n = \frac{\lambda+\mu+n}{n+1-\mu}$, $n \in \mathbb{Z}$.

If $R = R_{\lambda,\mu}$ is in the complementary series, then $T \in \{T_2, T_3\}$ by (6) and Remark 4.2. If

$$x_n = \sqrt{\frac{\Gamma(\lambda+\mu+n)}{\Gamma(1-\mu+n)}} f_n, \qquad n \in \mathbb{Z},$$

then x_n is an orthonormal basis of $\mathcal{H}^{\lambda,0}$ and in terms of this basis,

$$\begin{aligned}
T_2 x_n &= \sqrt{\frac{\Gamma(\lambda+\mu+n)}{\Gamma(1-\mu+n)}} T_2 f_n \\
&= \sqrt{\frac{\Gamma(\lambda+\mu+n)}{\Gamma(1-\mu+n)}} f_{n+1} \\
&= \sqrt{\frac{\Gamma(\lambda+\mu+n)}{\Gamma(1-\mu+n)}} \sqrt{\frac{\Gamma(2-\mu+n)}{\Gamma(\lambda+\mu+n+1)}} x_{n+1} \\
&= \sqrt{\frac{1-\mu+n}{\lambda+\mu+n}} x_n,
\end{aligned}$$

so the weight sequence is $w_n = \sqrt{\frac{1-\mu+n}{\lambda+\mu+n}}$, $n \in \mathbb{Z}$.

Now suppose $R = D^-_{2-\lambda} \oplus D^+_\lambda$ acting on $\mathcal{H} = \mathcal{H}^{(2-\lambda),0} \oplus \mathcal{H}^{\lambda,0} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (say). Let

$$g_n = \begin{cases} (f_{-1-n}, 0), & n < 0, \\ (0, f_n), & n \geq 0. \end{cases}$$

Then g_n, $n \in \mathbb{Z}$ is a basis of $\mathcal{H}$,

$$\{F \in \mathcal{H} \mid R(\exp(th))F = e^{-i(2n+\lambda)t} F\} = \mathbb{C} g_n, \qquad n \in \mathbb{Z},$$

and $T g_n = a_n g_{n+1}$ for some $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. For future reference, we record

$$R(e)g_n = \begin{cases} (1-\lambda-n)g_{n-1}, & n<0, \\ 0, & n=0, \\ -ng_{n-1}, & n>0, \end{cases}$$

(9)

$$R(f)g_n = \begin{cases} (n+1)g_{n+1}, & n<-1, \\ 0, & n=-1, \\ (\lambda+n)g_{n+1}, & n>-1. \end{cases}$$

Write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i,j \in \{1,2\}$, $T_{12} = 0$ and

$$T_{21}f_n = \begin{cases} 0 & n>0 \\ rf_0 & n=0, \end{cases}$$

for some $r \in \mathbb{C}$. So $D^-_{2-\lambda}$ is associated to T_{11} and D^+_λ is associated to T_{22}, and so

$$\begin{aligned} T_{11}f_0 &= 0, \\ T_{11}f_n &= \frac{n}{1-\lambda+n} f_{n-1}, \qquad n = 1,2,\dots, \\ T_{22}f_n &= f_{n+1}, \qquad n = 0,1,2,\dots. \end{aligned}$$

So

$$a_n = \begin{cases} \frac{(1+n)}{\lambda+n}, & n<-1, \\ r, & n=-1, \\ 1, & n>-1. \end{cases}$$

We claim that $\lambda = 1$. This requires the full strength of the condition that T is homogeneous (at the infinitesimal level). We compute

$$\begin{aligned} \kappa(L)T = & \left.\frac{d}{dt}\right|_{t=0} \kappa(\exp tL)T \\ = & \left.\frac{d}{dt}\right|_{t=0} \varphi^{-1}_{1,-\tanh t}(T) \\ = & \left.\frac{d}{dt}\right|_{t=0} \varphi_{1,\tanh t}(T) \\ = & \left.\frac{d}{dt}\right|_{t=0} \frac{T-(\tanh t)\mathrm{I}}{I-(\tanh t)T} \\ = & \, T^2 - \mathrm{I}, \end{aligned}$$

and

$$\begin{aligned}\kappa(M)T &= \left.\frac{d}{dt}\right|_{t=0} \kappa(\exp tM)T \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi_{1,-i\tanh t}^{-1}(T) \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi_{1,i\tanh t}(T) \\ &= \left.\frac{d}{dt}\right|_{t=0} \frac{T - i(\tanh t)\mathrm{I}}{I + (\tanh t)T} \\ &= -i(T^2 + \mathrm{I}).\end{aligned}$$

Therefore $\kappa(e)T = -\mathrm{I}$ and $\kappa(f)T = T^2$. Now, from (9) we see that $(\kappa(f)T)g_{-1} = [R(f), T]g_{-1} = r\lambda g_1$. But $T^2 g_{-1} = rg_1$. So $\lambda = 1$, proving the claim.

In this situation, $x_n = g_n$, $n \in \mathbb{Z}$ is already an orthonormal basis of $\mathcal{H}$. So the weight sequence is

$$w_n = \begin{cases} 1, & n < -1, \\ r, & n = -1, \\ 1, & n > -1. \end{cases}$$

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